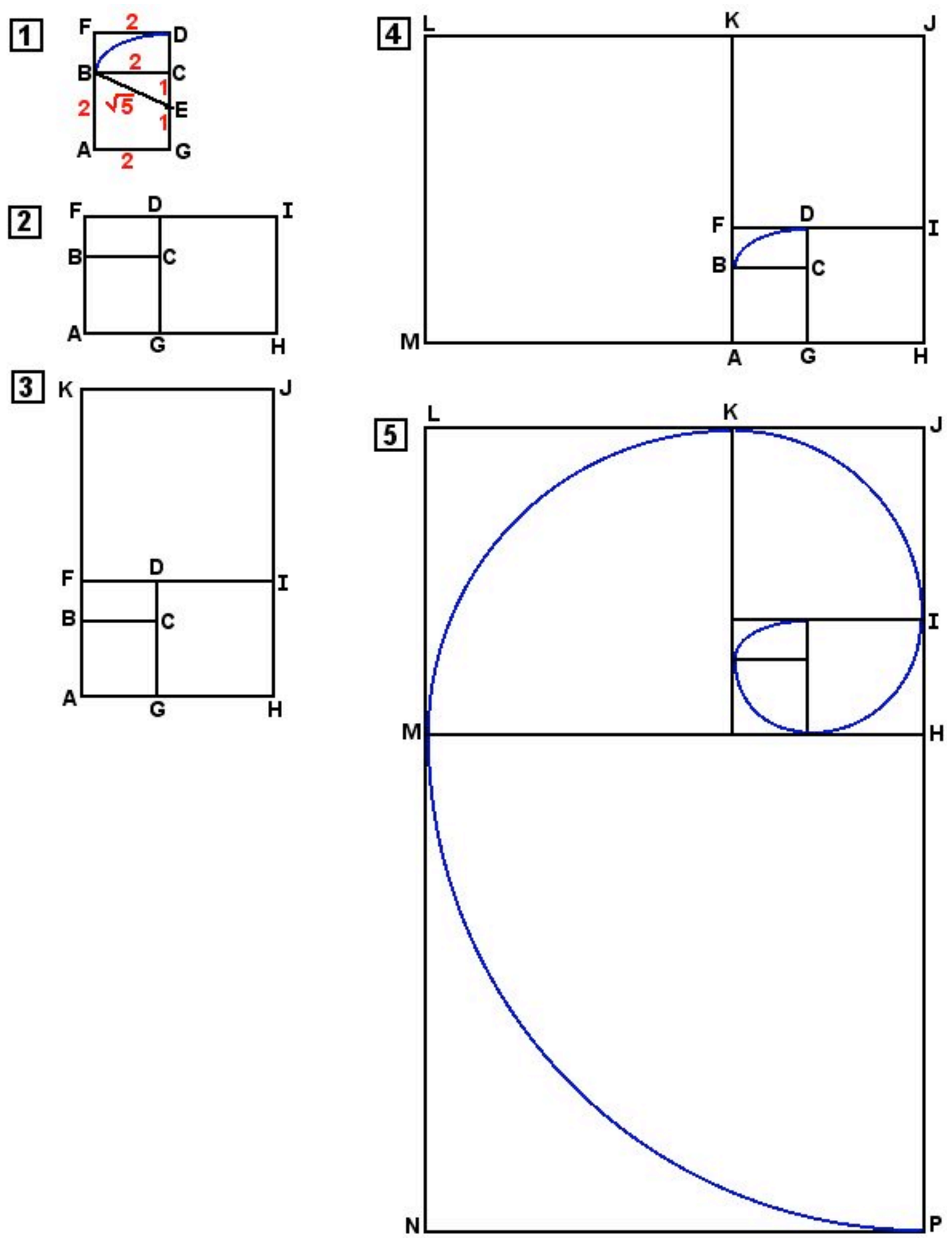
Rationalizing Denominators and the Golden Rectangle and Archimedean Spiral

If you were asked to work out whether $\frac{3+\sqrt{5}}{1+\sqrt{5}}$ and $\frac{1+\sqrt{5}}{2}$ were equal with a calculator you could simplify the first to get: 1.618033989… and if you simplify the second you get: 1.618033989…. Now these look the same, but because the decimals are rounded off to only 9 decimal places we do not really know since the numbers could start to differ on the tenth decimal place. So, at times, when I was teaching how to rationalize a denominator, I thought that maybe this is obsolete knowledge, and should be dropped. But now I realize that it has its uses when you want to have an EXACT answer as the following exercise will demonstrate.

When I stumbled on the picture on the right, of joined golden ratio rectangles and the Archimedean spiral that results from them, I soon bumped into rationalizing denominators again. In the diagram, number (5) on the lower right is the final answer.



Here is how it is built up, and where the golden ratio comes in. From the earlier work, we have seen that any ratio equal to $\frac{1+\sqrt{5}}{2}$, or 1.618033989 is the golden ratio "phi", labeled "Φ" We start, in diagram 1, with a 2 x 2 square ABCG, with E, the mid-point of CG. Thus CE = EG = 1. Now by putting a compass point at E, and radius of EB, draw an arc that meets at GC extended at D. Now extend AB to point F so that AGDF forms a rectangle. Since triangle BCE is a right triangle with sides of 2 and 1, the hypotenuse BE must equal $\sqrt{5}$. Therefore ED is also $\sqrt{5}$ and the length of the rectangle GD is then equal to 1 + $\sqrt{5}$. Thus the ratio of length over width of rectangle AGDF is $\frac{1+\sqrt{5}}{2}$ or Φ, the golden ratio.

Next, in diagram 2, I extend FD and AG to form a square GHID. Since the sides are all the same length as DG, they are all equal to 1 + $\sqrt{5}$. So the new rectangle formed AHIF has a length, FI, of 2 + 1 + $\sqrt{5}$, or 3 + $\sqrt{5}$. Its width is FA so that equals 1 + $\sqrt{5}$. Hence its length to width ratio is equal to $\frac{3+\sqrt{5}}{1+\sqrt{5}}$. Let's rationalize the denominator to see what this ratio simplifies to. See the next page.

$$\frac{(3+\sqrt{5})}{(1+\sqrt{5})}\times\frac{(1-\sqrt{5})}{(1-\sqrt{5})}=\frac{3-2\sqrt{5}-5}{1-5}=\frac{-2-2\sqrt{5}}{-4}=\frac{-2(1+\sqrt{5})}{-2(2)}=\frac{1+\sqrt{5}}{2}$$

Wow, we get the ratio simplifying to Φ, the golden ratio. Now let's attach another FIJK onto what we have. This is diagram number 3 from the image on page 1. Its sides are all equal to $3+\sqrt{5}$. So the new rectangle AHJK has a length, HJ, of $3+\sqrt{5}+1+\sqrt{5}$ or $4+2\sqrt{5}$. Its width AH is $2+1+\sqrt{5}$ or $3+\sqrt{5}$. Its length ot width ratio is $\frac{4+2\sqrt{5}}{3+\sqrt{5}}$. Is it also a golden rectangle? Let's simplify it by rationalizing its denominator.

$$\frac{(4+2\sqrt{5})}{(3+\sqrt{5})}\times\frac{(3-\sqrt{5})}{(3-\sqrt{5})}=\frac{12+2\sqrt{5}-10}{9-5}=\frac{2+2\sqrt{5}}{4}=\frac{2(1+\sqrt{5})}{2(2)}=\frac{1+\sqrt{5}}{2}$$

Another golden rectangle! Now let's attach another KLMA onto what we have. This is diagram number 4 from the image on page 1. Its sides are all equal to $4+2\sqrt{5}$. Therefore the new rectangle HJLM has a length equal to $4+2\sqrt{5}+3+1\sqrt{5}$, or $7+3\sqrt{5}$. Its width, LM is $4+2\sqrt{5}$, therefore its length to width ratio is $\frac{7+3\sqrt{5}}{4+2\sqrt{5}}$. Is it also a golden rectangle? Let's simplify it by rationalizing its denominator.

$$\frac{(7+3\sqrt{5})}{(4+2\sqrt{5})}\times\frac{(4-2\sqrt{5})}{(4-2\sqrt{5})}=\frac{28-2\sqrt{5}-30}{16-20}=\frac{-2-2\sqrt{5}}{-4}=\frac{-2(1+\sqrt{5})}{-2(2)}=\frac{1+\sqrt{5}}{2}$$

Another golden rectangle!. Finally I added, in diagram (5), the square MNPH, with sides of $7+3\sqrt{5}$ each. This forms a new rectangle JLNP with a length. LN, of $4+2\sqrt{5}+7+3\sqrt{5}$, of $11+5\sqrt{5}$. Its width, LJ is $7+3\sqrt{5}$, so its length to width ratio is $\frac{11+5\sqrt{5}}{7+3\sqrt{5}}$. Is it also a golden rectangle? Let's simplify it by rationalizing its denominator.

$$\frac{(11+5\sqrt{5})}{(7+3\sqrt{5})}\times\frac{(7-3\sqrt{5})}{(7-3\sqrt{5})}=\frac{77+2\sqrt{5}-75}{49-45}=\frac{2+2\sqrt{5}}{4}=\frac{2(1+\sqrt{5})}{2(2)}=\frac{1+\sqrt{5}}{2}$$

Again, a golden rectangle. Finally, another pattern also emerges by looking at the successive sides of the squares added. Observe the sequence below:

2, $1+\sqrt{5}$, $3+\sqrt{5}$, $4+2\sqrt{5}$, $7+3\sqrt{5}$, $11+5\sqrt{5}$. It's a Fibonacci Sequence! Thus, the golden ratio emerges as the ratio between the terms as we have seen with all Fibonacci Sequences.