Sums of Geometric Series

Just like arithmetic sequences, geometric sequences become a series when the terms are joined with a plus or minus sign. Here are some examples:

1 + 2 + 4 + 8 + 16 + 32 + 64

81 + 27 + 9 + 3 + 1 + 1/3 + …

1 – 0.8 + 0.64 – 0.512 + 0.4096 - …

The first example is a finite series since it comes to an end, while the second and third examples are of infinite series since they go on to an infinite number of terms.

If your variables are: "t_1 = a" = first term. "n" = number of terms. " $t_n = \ell$ " = last term. " r " = ratio between terms. If the ratio is unknown it can be found by dividing any term by the term before it.
Then the three formulas that we will use for finding sums of series are:

(1) $S_n = \frac{t_1(1-r^n)}{1-r}$ This is used if the last term (t_n) is NOT known. Alternate form: $S_n = \frac{a(1-r^n)}{1-r}$

(2) $S_n = \frac{t_1 - t_n r}{1-r}$ This is used if the last term (t_n) is known. Alternate form: $S_n = \frac{a - \ell r}{1-r}$

(3) $S_n = \frac{t_1}{1-r}$ This is used for the sum of an infinite series. Alternate form: $S_n = \frac{a}{1-r}$

Note, the third formula is merely the second formula with the final (or infinite term number) equaling zero. This will happen only if $|r| < 1$. So the ratio must be a positive or negative fraction less than one.

Here are some examples of them being used.

Observe the following table where we take the terms and their sums:

Term number	**Term**	**Sum up to this term**
1	**1**	**1**
2	**2**	**3**
3	**4**	**7**
4	**8**	**15**
5	**16**	**31**
6	**32**	**63**
7	**64**	**127**
8	**128**	**255**

As you may have noticed, the sum equals one less than then next term. The proof of why this works is as follows: Since the term is known, then the sum to any term is: $S_n = \frac{t_1 - t_n r}{1-r}$

This is equal to the sum up to the term before it PLUS the term itself, or:

$$S_{n-1} = \frac{t_1 - t_{n-1}r}{1-r} + t_n =$$

$$\frac{t_1 - t_{n-1}r}{1-r} + \frac{t_n(1-r)}{1-r} =$$

$$\frac{t_1 - t_{n-1}r + t_n(1-r)}{1-r} =$$

$$\frac{t_1 - t_{n-1}r + t_n - t_n r)}{1-r}\text{, but since } t_{n-1}r = t_n\text{, we have:}$$

$$\frac{t_1 - t_n + t_n - t_n r)}{1-r} = \frac{t_1 - t_n r}{1-r}$$

Example (2) Find the sum of the first 15 terms of: 1 + 1.5 + 2.25 + 3.375 + …

Step (1) is to write out the variables: $t_1 = 1$; n = 15; t_n = ??? ; and r = 1.5. Since we do NOT know the last term, then we use this formula: $S_n = \frac{t_1(1-r^n)}{1-r} = \frac{1(1-1.5^{15})}{1-1.5} = 873.788$

Your calculator should look like this: **1(1 – 1.5^15) / (1 – 1.5) = 873.788**

Example (3) Find the sum of: 1 + 4 + 16 + … + 1 048 576.

Step (1) is to write out the variables: $t_1 = 1$; n = ??? ; t_n = 1 048 576 ; and r = 4. Since we do NOT know the number of terms, then we use this formula: $t_n = t_1 r^{n-1}$ first to find the number of terms.

$$1048576 = 1(4)^{n-1}$$

$$4^{10} = 4^{n-1}\text{, and } \therefore\ 10 = n-1$$

$$10 + 1 = n = 11$$

So there are 11 terms, now use $S_n = \frac{t_1 - t_n r}{1-r}$, so

$$S_{11} = \frac{1 - 1048576(4)}{1-4} = 1{,}398{,}101$$

Example (4) If I dropped a ball from 8 metres in height, and if the ball rebounded perfectly to one-half of its dropped height each time, how far would the ball travel if it continued to infinity. Here is a look at the problem:

8 + 4 + 4 + 2 + 2 + 1 + 1 + 0.5 + 0.5 + 0.25 + 0.25 + … or 8 + 8 + 4 + 2 + 1 + 0.5 + …

This equals 8 + (and infinite series with $t_1 = 8$; and r = 0.5).

The sum of this infinite series is: $S_n = \frac{8}{1-0.5} = \frac{8}{0.5} = 16$

So, the final length of the path is 8 + 16 = 24 metres.