Sums of Arithmetic Series

When a sequence such as 1, 3, 5, 7, 9, … is joined with plus signs we get the series: 1 + 3 + 5 + 7 + 9 + …

Before we see the formulas for working out the sum of an arithmetic sequence, a little "math bedtime story". (your parents did read "math bedtime stories to you didn't they"?)

Johann Karl Friedrich Gauss, a German Mathematician, born April 30, 1777, died February 23, 1855

Son of a gardener, was a prodigy capable of great feats of memory and mental calculations. At the age of three, he corrected his father's calculations for a payroll sheet. At the age of ten, his teacher J. G. Burtner, felt his class was a bit rowdy and asked them to takeout their slates and find the sum of the first one hundred numbers. Guass looked at the problem: 1 + 2 + 3 + 4 + … + 97 + 98 + 99 + 100, and soon realized that he could form pairs 1 + 100, 2 + 99, 3 + 98, 4 + 97, etc that all had the sum of 101. Since there were 50 pairs, the answer must be 101 x 50 or 5 050.

So, what he asked to do was find the sum of the arithmetic series: 1 + 2 + 3 + 4 + … + 97 + 98 + 99 + 100. The two formulas to find the sums are as follows:

$s_n = \dfrac{n(t_1 + t_n)}{2}$ is used to find the sum (S_n) when the first term (t_1) and the last term (t_n) are known.

Sometimes the first term is labeled "**a**" and the last term is labeled "ℓ " in which case the formula becomes:

$s_n = \dfrac{n(a + \ell)}{2}$. If you do not know the last term, then the formula is: $s_n = \dfrac{n(2t_1 + (n-1)d)}{2}$

Using this, let's work out Gauss' problem above: $t_1 = 1$, $d = 1$, $t_n = 100$ and $n = 100$

$s_n = \dfrac{100(1+100)}{2} = \dfrac{100}{2}(101) = 50 \times 101 = 5050$. So you can see we do have 50 x 101 show up in the working. I often wonder if Gauss found the pattern on the fly (which makes for a good story) OR whether he was aware of the formula already (which is still a pretty good story, since he was only 10 at the time).

Let's use the other formula to "prove" a conjecture. Observe the table below

Term Number	Term	Sum up to this term
1	1	1
2	3	4
3	5	9
4	7	16
5	9	25
6	11	36
7	13	49
8	15	64
9	17	81
10	19	100

With a bit of observation, you can see the sum of the first 6 terms is 6^2, first 7 terms is 7^2, etc. So my conjecture is that **"the sum of the first "n" odd numbers is n^2."** The proof follows on the next page.

$t_1 = 1$, $d = 2$, $n = n$, so:

$$s_n = \frac{n(2(1)+(n-1)2)}{2} = \frac{n(2+2n-2)}{2} = \frac{n(2n)}{2} = \frac{2n^2}{2} = n^2$$

So, sure enough, the sum is always n^2. QED

Here is another example: Find the sum of the following series: - 13 – 8 – 3 + 2 + 7 + … + 122

First step, is to label your variables: $t_1 = -13$, $d = 5$, $n = ??$ $t_n = 122$. So we don't have enough information yet to fill in either formula since we do NOT know what "n" is equal to. Use the formula from last week to find "n":

$t_n = t_1 + (n-1)d$ becomes: $122 = -13 + (n-1)5$. Add 13 to both sides gets us $135 = 5(n-1)$.

Divide both sides by 5 and we get $27 = n - 1$, and now add "1" to both sides and we get: $n = 28$. Now we can use the formulas above, I'll use the easiest one:

$$s_{28} = \frac{28(-13+122)}{2} = \frac{28}{2}(109) = 14(109) = 1{,}526$$

Next weeks, geometric sequences and series.