Geometric Sequences

If you multiply or divide by the same thing each time in between terms, then we have what is called a geometric sequence. The number you are multiplying or dividing by is called the "ratio". Here are some examples:

1, 2, 4, 8, 16, 32, 64, … ratio = 2

3, 9, 27, 81, 243, 729, … ratio = 3

100, 10, 1, 0.1, 0.01, … ratio = $\frac{1}{10}$, **or** 0.1

+ 16, – 8, + 4, – 2, + 1, – 0.5, … ration = $-\frac{1}{2}$, **or** -0.5

If you cannot easily see the ratio, it can always be found by dividing any term by the term in front of it. So, in the last example above, we have:

$\frac{-8}{16} = -0.5$, **or** $\frac{+4}{-8} = -0.5$, **or** $\frac{-2}{4} = -0.5$, **or** $\frac{+1}{-2} = -0.5$, **or** $\frac{-0.5}{+1} = -0.5$, **etc**

Just like with arithmetic sequences, we will start with a formula that will give the value of any term:

$$t_n = t_1(r)^{n-1},$$

where "$\mathbf{t_n}$" is the "nth or last" term, "$\mathbf{t_1}$" = the first term (sometimes labeled "a"), "**r**" = the ratio and "**n**" = the number of terms in the sequence.

Here are some sample problems:

(1) Find the 20th term in fhe first sequence above: 1, 2, 4, 8, 16, 32, 64, …
First step is to list your variables: $\mathbf{t_1 = 1}$ **(or a = 1), r = 2, n = 20,** $\mathbf{t_n = ?}$
Second step is to fill in the formula with what you know:

$$t_n = t_1(r)^{n-1}$$

$$t_{20} = 1(2)^{20-1} = 2^{19} = 524{,}288$$

(2) Find the number of terms in this sequence: 3, 9, 27, 81, …, 19 683, 59 049
First step is to list your variables: $\mathbf{t_1 = 3}$ **(or a = 3), r = 3, n = ???,** $\mathbf{t_n = 59\ 049}$
Second step is to fill in the formula with what you know:

$$t_n = t_1(r)^{n-1}$$

$$59{,}049 = 3(3)^{n-1}$$

$$\frac{59049}{3} = 3^{n-1} \textbf{ or } 19683 = 3^{n-1}$$

At this point you can solve it by using exponents, or by using logarithms. Both solutions are below on the next page:

$$19683 = 3^{n-1}$$
$$3^9 = 3^{n-1}, \therefore 9 = n - 1$$
$$9 + 1 = n, \text{ so } n = 10$$

$$19683 = 3^{n-1}$$
$$\log(19683) = \log(3^{n-1})$$
$$\log(19683) = (n-1)\log(3)$$
$$\frac{\log(19683)}{\log(3)} = n - 1$$
$$9 = n - 1, \text{ so } 9 + 1 = n, \text{ or } n = 10$$

Whichever method you use, the number of terms is 10. Note that logarithms would have to be used if the number of the left (which was 19 683) was NOT a power of 3.

(3) Find the first term in this sequence: : $t_6 = 34$ **and** : $t_{12} = 2\,587$

For this type of problem I usually draw a little diagram like the one below:

___,	___,	___,	___,	___,	**34**,	___,	___,	___,	___,	___,	**2 587**
t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}	t_{11}	t_{12}
					t_1	t_2	t_3	t_4	t_5	t_6	t_7

If I start to fill in the variables with $t_6 = 34$ and $t_{12} = 2\,587$, then I have too many values, so what I do is shift the term numbers to what you see in red above. Now using the red term numbers I have:

List your variables: **$t_1 = 34$ (or $a = 34$), $r = ??$, $n = 7$, $t_7 = 2\,587$**

Next step is to fill in the formula with what you know:

$$t_7 = 34(r)^{7-1}$$
$$2587 = 34(r)^{7-1}, \text{ now divide both sides by } 34$$
$$\frac{2587}{34} = r^6, \text{ now take the 6th root of both sides}$$
$$\sqrt[6]{\frac{2587}{34}} = \sqrt[6]{r^6}, \text{ and simplifying, we get:}$$
$$\sqrt[6]{76.08823529} = r, \text{ or } r = 2.058509704$$

Now dividing 34 by 2.058509704 five times gives you the first term of: 0.9198

(4) Find the 4 geometric means between 25 and 356.

Here the diagram looks like:

25,	___,	___,	___,	___,	**356**
t_1	t_2	t_3	t_4	t_5	t_6

List your variables: **$t_1 = 25$ (or $a = 25$), $r = ??$, $n = 6$, $t_7 = 356$**

Next step is to fill in the formula with what you know. This is done on the following page:

$$t_6 = 25(r)^{6-1}$$

$$356 = 34(r)^{6-1}, \text{ now divide both sides by 34}$$

$$\frac{356}{34} = r^5, \text{ now take the 5th root of both sides}$$

$$\sqrt[5]{\frac{356}{25}} = \sqrt[5]{r^5}, \text{ and simplifying, we get:}$$

$$\sqrt[5]{14.24} = r, \text{ or } r = 1.70099093, \text{ or } r \approx 1.7$$

Now, since "r" is approximately = 1.7, I can multiply 25 by 1.7 and each term after that, by 1.7 to fill in the blanks as you see below:

25,	42.5 ,	72.25,	122.825,	208.8025,	356
t_1	t_2	t_3	t_4	t_5	t_6

Thus, the four geometric means between 25 and 356 are as you see in red above.

If you take the same problem, but ask for 11 geometric means between 1 and 2 you get:

1 ,	___,	___,	___ ,	___,	___,	___,	___,	___,	___,	___,	___,	2
t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}	t_{11}	t_{12}	t_{13}

Working the formula with **t_1 = 1 (or a = 1), r = ?? , n = 13, t_{13} = 2, we get:**

$$t_{13} = 1(r)^{13-1}$$

$$2 = 1(r)^{13-1}, \text{ now divide both sides by 34}$$

$$\frac{2}{1} = r^{12}, \text{ now take the 5th root of both sides}$$

$$\sqrt[12]{2} = \sqrt[12]{r^{12}}, \text{ and simplifying, we get:}$$

$$\sqrt[12]{2} = r, \text{ or } r = 1.059463094$$

This particular geometric sequence is the musical scale, where if the key "A" has a frequency of "1", then the octave above the key, which is the next "A" key, has a frequency of 2. The ratio between all the notes, including the black notes, is then 1.059463094.

If you have a population of 34 897 in the year 2009, and a population growth of 2.5%, then the population in the year 2025 is a geometric sequence problem with **t_1 = 34 897, r = 1+2.5% or 1.025 , n = 17, t_{13} = ???**

So, as you can see there are some really useful geometric sequences out there. Next week, we will look at the sums of geometric series to finish off this section.