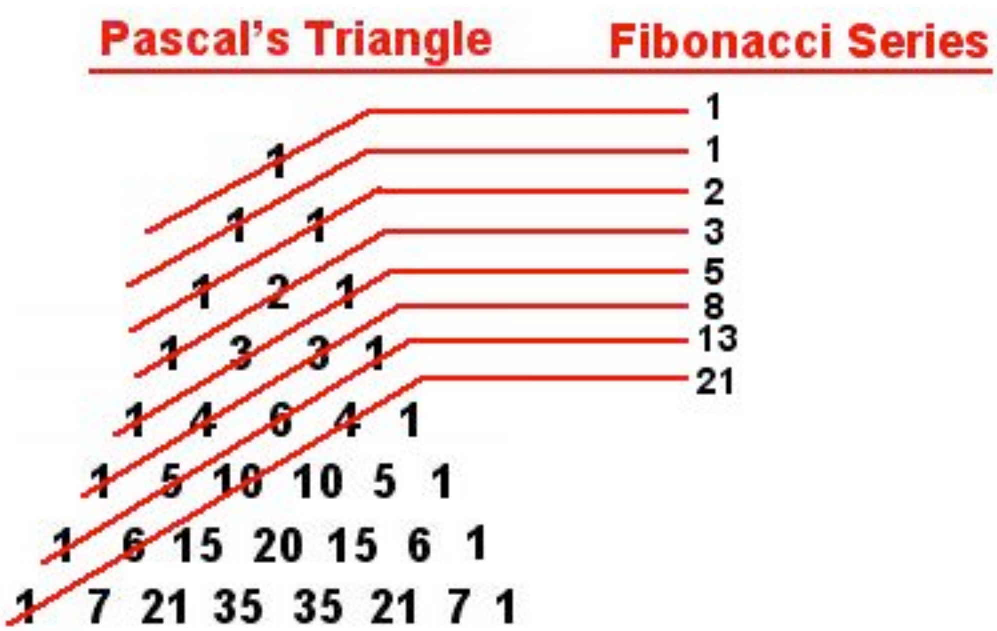
Applications of Pascal's Triangle

Below is Pascal's Triangle to the 7th row.

Row	Pascal's Triangle	Sum	Power of 2
0	1	1	2^0
1	1 1	2	2^1
2	1 2 1	4	2^2
3	1 3 3 1	8	2^3
4	1 4 6 4 1	16	2^4
5	1 5 10 10 5 1	32	2^5
6	1 6 15 20 15 6 1	64	2^6
7	1 7 21 35 35 21 7 1	128	2^7

Add 2 numbers above left and right $\text{Sum} = 2^{\text{Row}}$

First of all, lets look at some of the patterns involved. To generate each row, start and end with a "1", and for each new cell, add the numbers that appear above left and above right. The sum of each row equals 2^{ROW}, so the 8th row would have a sum of $2^8 = 256$. As you can also see, the second number in each row is the same as the row number. The third number in each row, starting at row 2, is a triangular number. Also, the number in each row are symmetric about a vertical line. Looking at the diagonals below:



You can see that the Fibonacci Sequence of: 1, 1, 2, 3, 4, 13, 21 appears. Since the ratio of each term divided by the term in front of it approaches the Golden Ratio of 1.618033989… that is hidden in the triangle as well. Okay, so much for patterns, but what can you do with it?

(1) **Pascal's Triangle and Combinations**: Each term of Pascal's Triangle can be found using the combination formula: ${}_nC_r = \frac{n!}{(n-r)!r!}$. Thus it shows up in probability and combinations. For instance, four people shake hands with each of the other people, how many handshakes are there? Answer: ${}_4C_2 = 6$. How many games will there be if there are 7 teams playing a round-robin tournament where each team plays each other team just once? Answer: ${}_7C_2 = 21$. As you can see these are just two of the triangular numbers found as the 3rd number in each line of Pascal's Triangle. Now these don't have to be handshakes, they could be glasses clinked at a toast, lines of communication in an organization, airplane flights between airports, and all sorts of network problems. By the way, there is a short cut to answer these: take the number of people (or teams etc) and multiply by the number one less than that, and the divide by two.
So with 7 people we do: **7 x 6 ÷ 2 = 21**.

(2) **Pascal's Triangle and Probability**: Fred and Mary have 6 children, 3 boys and 3 girls. What is the probability of that happening? Pascal's Triangle can solve that for probabilities that are 50 – 50. Just write down all seven numbers from row six: 1 6 15 20 15 6 1. Now add "gb" after each:

1 g b , 6 g b , 15 g b , 20 g b , 15 g b , 6 g b , 1 g b Now start with an exponent of 6 on the **g** and 0 on the **b** with the first term. Now for each term count the exponents on the g's down until reaching 0 and the exponents on the b's up until reaching 6:

$\mathbf{1\,g^6\,b^0,\ 6\,g^5\,b^1,\ 15\,g^4\,b^2,\ 20\,g^3\,b^3,\ 15\,g^2\,b^4,\ 6\,g^1\,b^5,\ 1\,g^0\,b^6}$ Now place the total of this row of $2^6 = 64$ under each of the digits:

$$\frac{1}{64}g^6b^0, \frac{6}{64}g^5b^1, \frac{15}{64}g^4b^2, \frac{20}{64}g^3b^3, \frac{15}{64}g^2b^4, \frac{6}{64}g^1b^5, \frac{1}{64}g^0b^{06}$$

These give you all the probabilities involving 6 children (or flipping 6 coins). To answer or question, look for 3 girls and 3 boys by finding the $\mathbf{g^3\,b^3}$ term. The fraction in front is the probability of that happening, so the answer is $\frac{20}{64} = \frac{5}{16} = 0.3125 = 31.25\ \%$. So the probability of 1 girl and 5 boys? Answer: Look for the $\mathbf{g^1\,b^5}$ term and choose the number in front of it: $\frac{6}{64} = \frac{3}{32} = 0.09375 = 9.375\ \%$

(3) **Pascal's Triangle and the Binomial Expansion**: What is $(a + b)^5$? We approach this just like we did above, but put a "+" sign in instead of a comma between each term, and do not put in a denominator.

$$(a + b)^5 = 1\,a^5\,b^0 + 5\,a^4\,b^1 + 10\,a^3\,b^2 + 10\,a^2\,b^3 + 5\,a^1\,b^4 + 1\,a^0\,b^5$$

and since $\mathbf{a^0 = b^0 = 1}$ we can simplify to:

$$(a + b)^5 = 1\,a^5 + 5\,a^4\,b + 10\,a^3\,b^2 + 10\,a^2\,b^3 + 5\,a\,b^4 + 1\,b^5$$

(4) **Pascal's Triangle and the Binomial Expansion AND Basketball:** Steve Nash shoots 7 three-point shots, what is the probability that he makes 5 out of the 7, if the probability of success is 42% ? Use the 7th row of Pascal's Triangle: I have used "y" for "yes" he made it, and "n" for "no" he missed.

$$(y + n)^7 = 1\,y^7\,n^0 + 7\,y^6\,n^1 + 21\,y^5\,n^2 + 35\,y^4\,n^3 + 35\,y^3\,n^4 + 21\,y^2\,n^5 + 7\,y^1\,n^6 + 1\,y^0\,n^7$$

Now let each y = 0.42 and each n = 0.58 (which is 1 – 0.42).

$\mathbf{1\,(.42)^7\,(.58)^0 + 7\,(.42)^6\,(.58)^1 + 21\,(.42)^5\,(.58)^2 + 35\,(.42)^4\,(.58)^3 + 35\,(.42)^3\,(.58)^4 + 21\,(.42)^2\,(.58)^5 +}$
$\mathbf{7\,(.42)^1\,(.58)^6 + 1\,(.42)^0\,(.58)^7}$. So 5 "Yess's" and 2 "No's", has a probability of $\mathbf{21\,(.42)^5\,(.58)^2 = 0.092326}$.

I'll add some shortcuts and some more to this.