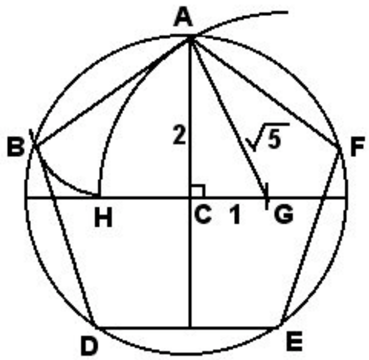
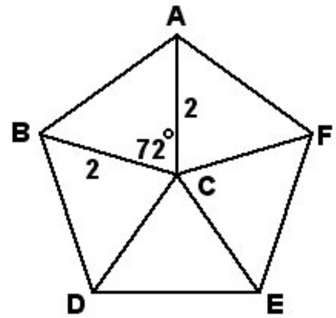
Why the Construction of a Regular Pentagon Works

In the diagram below on the left, you will see the construction of the regular pentagon. Remember the adjective "regular" means that all sides and all angles are equal.





Let's look at the steps we took to construct this pentagon. Let's assume the radius of the circle is equal to 2.

(1) Construct the Perpendicular Bisector of a diameter to form a right angle ∠ACG
(2) Bisect the radius to find point "G", therefore CG = 1, and since AC = 2, by Pythagoras we can find that the length AG = $\sqrt{5}$.
(3) Place your compass point at "G" and radius of GA, draw an arc from A to H. Thus AH = $\sqrt{5}$ and therefore HC = $\sqrt{5}-1$
(4) Now with your compass point at "A", and radius of AH, draw an arc from H to B. This gives you the length of each side of the pentagon.
(5) Make AB = BD = DE = EF. Check that FA is equal to these also. Join up A to B to D to E to F to A and you have your pentagon.

Thus we have all sides equal to AH. Since triangle AHG is a right triangle with HC = $\sqrt{5}-1$ and AC = 2, then using Pythagoras we can compute AH (and all the sides of the pentagon) this way:

$$\boldsymbol{AH}^2 = (\sqrt{5}-1)^2 + 2^2$$

$$\boldsymbol{AH}^2 = 5.527864045$$

$$\boldsymbol{AH} = \sqrt{5.527864045} \rightarrow \boldsymbol{AH} = 2.351141009$$

So each side of the pentagon is equal to 2.351141009. Now looking at the right hand diagram above, if we have a regular pentagon then the central angle ∠ACB = 360° divided by 5 or 72°. The radii are both equal to 2, so the length AB, using the Law of the Cosines is:

$$\boldsymbol{AB}^2 = 2^2 + 2^2 - 2(2)(2)\cos 72^{\circ}$$

$$\boldsymbol{AB}^2 = 5.527864045$$

$$\boldsymbol{AB} = \sqrt{5.527864045} = 2.351141009$$

Thus, it works. Now to show that it works for any radii, we should redo the proof with radius = "r" not "2". I will leave that proof until later.

As an "extra" though, let's use the right hand diagram and see what the distance BF is compared to AB. The formula for the sum of all the interior angles of a polygon with sides of length "n" is Sum = 180(n – 2). So the sum of all the angles of a pentagon is 180 (5 – 2) = 540°. Thus each angle is 540° divided by 5 or 108°. On the next page I will show you, using the Law of the Cosines again, what the ratio of BF is compared to AB is equal to.

$$BF^2 = AB^2 + AF^2 - 2(AB)(AF)\cos 108^o$$

But AB = AF, so

$$BF^2 = AB^2 + AB^2 - 2(AB)(AB)\cos 108^o$$

$$BF^2 = 2AB^2 - 2AB^2\cos 108^o$$

$$BF^2 = 2AB^2(1 - \cos 108^o)$$

$$\frac{BF^2}{AB^2} = 2(1 - \cos 108^o) = 2.618033989$$

Hmmm, something about the number to the right of the decimal point seems familiar...

$$\sqrt{\frac{BF^2}{AB^2}} = \sqrt{2.618033989} = 1.618033989$$

$$\frac{BF}{AB} = 1.618033989$$

Ahaha, the ratio = ϕ, the Golden ratio

And that is another thing about this magic number, its square is just 1 more than it. If you solve this equation: $x^2 = x + 1$, you'll find that x = 1.618033989 is the only positive solution, the other solution is – 0.618033989 or the negative of the decimal part of the golden ratio.