

## Pythagoras, Music and Triangles

Greek Philosopher, Born c. 582 BC, Died c 497 BC

Pythagoras is reputed to have traveled widely, Egypt, the East, and so on. He left Samos in 529 BC and went to Croton, southern Italy. Here Pythagoras east-Greek traditions and founded a cult marked by secrecy and mysticism. The Pythagoreans, as he and his followers were called, were different than other cults in their interest in Mathematics and Astronomy. It is hard to pinpoint who of the Pythagoreans were responsible for his work, but for sake of brevity, I'll just refer to the work as coming from Pythagoras.

Pythagoras, of course, is most well know for his work with the right triangle. I'll talk about that later in this article, including a simple proof. However, to begin with I would like to talk about his contribution to music since that is the "theme" of this week.

Pythagoras discovered that strings of musical instruments would have a higher pitch if they were shorter. As well as that, he found that the ratio of the lengths of the strings were simple whole number ratios for the musical scale. The ratio, 1 : 2, gives you the octave; the ratio, 2 : 3, gives you the fifth; the ratio, 3 : 4, gives you the fourth. These ratios have been illustrated in the past three days, Monday, May 11, to Wednesday, May 13. Below I will show you the ratios on the white keys of the keyboard.

<b>C</b>	<b>D</b>	<b>E</b>	<b>F</b>	<b>G</b>	<b>A</b>	<b>B</b>	<b>C</b>
<b>4</b>	<b>4<math>\frac{1}{2}</math></b>	<b>5</b>	<b>5<math>\frac{1}{3}</math></b>	<b>6</b>	<b>6<math>\frac{2}{3}</math></b>	<b>7<math>\frac{1}{2}</math></b>	<b>8</b>

Alternate ratios (the above times 66): **264, 297, 330, 352, 396, 440, 495, 528**

So the ratio of C : G (the fifth) is 4 : 6, or 2 : 3. The ratio of the fourth is C : F and it is worked out below:

$$\frac{5\frac{1}{3}}{4} = \frac{\frac{16}{3}}{4} = \frac{16}{3 \times 3} = \frac{16}{12} = \frac{4}{3}$$

So the ratio of the fourth is 3 : 4.

Notice that another fourth is G : C (on the right) which is 6 : 8 or, of course, 3 : 4.

An interesting activity here is to take a single guitar string, and play the scale just on that one string. Then take the measurements of the frets from the base, and this gives you the string length for any note. Now, find the ratios, and see if you get near to the ratios above. To see whether two ratios are equal, or which is smaller, or larger, cross multiply and put the result of top left times bottom right on the left hand side of your answer, and bottom left and top right on the right hand side of your answer. Compare the results (<, =, >), and you get the relationship between the original ratios. Observe:

### Example 1

$$\frac{5}{4} ? \frac{330}{264}, \text{ cross - multiply}$$

$$5 \times 264 ? 330 \times 4$$

$$1320 = 1320, \text{ hence the ratios are equal}$$

$$\text{and } \frac{5}{4} = \frac{330}{264}$$

### Example 2

$$\frac{15}{24} ? \frac{33}{52}, \text{ cross - multiply}$$

$$15 \times 52 ? 33 \times 24$$

$$780 < 792, \text{ hence the ratios aren't equal}$$

$$\text{and } \frac{15}{24} < \frac{33}{52}$$

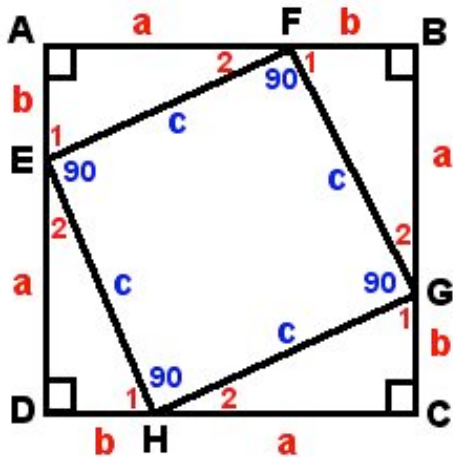
This led Pythagoras and his followers to believe that the whole universe could be explained by whole numbers and their ratios, the rational numbers, we call fractions. They gave these numbers all sorts of mystical properties. However, as you will see below, a problem occurred when working with a right triangle. On Friday, March 11<sup>th</sup>, we looked at Pythagorean Triples. Here are a few of them:

3 : 4 : 5, 5 : 12 : 13, 7 : 24 : 25, 8 : 15 : 17, and 17 : 114 : 145

Pythagoras and his followers discovered that these all fell into the relationship we now know as the Pythagorean Theorem: If the shorter two sides are labeled “a” and “b”, and the long side (the hypotenuse) labeled “c”, then:

$a^2 + b^2 = c^2$ , for example  $3^2 + 4^2 = 5^2$ , since  $9 + 16 = 25$ . Whole numbers that solve the Pythagorean theorem are called Pythagorean Triples. Another relationship has come up, I first noticed it and proved it about 1985, and I have never seen it since. I called it “Tetra’s Theorem” and have written about it and showed its proof on Friday, March 6<sup>th</sup>, I urge you to go and look at it.

Now look at the diagram below, and the simple proof that follows:



All four triangles are equal, since they each have two equal sides, “a” and “b”, and a right angle, (SAS). Since angles 1 plus 2 plus 90° = 180°, then angles 1 + 2 = 90°. This means each of the angles of the quadrilateral EFGH is equal to 90°, and each side is equal to “c”, then EFGH must be a square. Finally, each of the sides of square ABCD equals a + b. So,

Area of square ABCD equals area of each of the four triangles plus area of the square EFGH, thus:

$$(a + b)^2 = 4\left(\frac{1}{2} ab\right) + c^2$$

$$a^2 + 2ab + b^2 = 2ab + c^2$$

**and subtract 2ab from each side leaves you with :**

$$a^2 + b^2 = c^2$$

Thus, a proof for the Pythagorean Theorem. There are hundreds of different proofs for this theorem, apparently even one by Napoleon, and one by President James A Garfield of the USA. The following website: <http://www.cut-the-knot.org/pythagoras/> has some 81 proofs of Pythagoras. (As an aside, before I was adopted, my biological middle name was Garfield, after that president. I guess I come by my love of Mathematics from that!!)

As an end to the Pythagorean saga, they had trouble with the right triangle with sides of 1 and 1. According to their formula  $1^2 + 1^2 = c^2$  and therefore  $2 = c^2$ , and  $c = \sqrt{2}$ . The Pythagoreans were floored to find out that no whole number ratio (a rational number) could be found (and still hasn’t been found) that satisfies the square root of 2. Hence, they bumped into the irrational numbers, and their cult began to wane.