

## Math and the Musical Scale Part Three

At this point, after the last two days (Monday, May 11 and Tuesday, May 12) we have developed the C major scale (the white keys on a keyboard) with the ratios as we see below.

<b>C</b>	<b>D</b>	<b>E</b>	<b>F</b>	<b>G</b>	<b>A</b>	<b>B</b>	<b>C</b>
4	$4\frac{1}{2}$	5	$5\frac{1}{3}$	6	$6\frac{2}{3}$	$7\frac{1}{2}$	8

Now, if we read these ratios as frequencies, they are too low for the human ear to hear. Thus we will multiply all terms of the ratios by 66 to move this scale up to the middle of the piano.

(4, 4.5, 5, 5.3333, 6, 6.666667, 7.5, 8) all times 66 gives us the following:

(C) 264, (D) 297, (E) 330, (F) 352, (G) 396, (A) 440, (B) 495, (C) 528 – at least we now have all whole numbers (for now). This allows a frequency of the “A” key to be 440, since musicians prefer to tune the “A” at that frequency.

Now, if we look at the ratios of each note’s frequency with that of the note before it, we get:

Note	Frequency	Ratio	Round to 2	Decimal	Gap
C	264	1	1	0	
D	297	1.125	1.13	0.13	whole
E	330	1.111111111	1.11	0.11	whole
F	352	1.066666667	1.07	0.07	half
G	396	1.125	1.13	0.13	whole
A	440	1.111111111	1.11	0.11	whole
B	495	1.125	1.13	0.13	whole
C	528	1.066666667	1.07	0.07	half

As you can see above, there seems to be some gaps that are roughly twice the size of other gaps. In other words, there are notes missing. These are the black keys on the piano. We want to be able to start any triad, or scale, on any note, thus we need to put those keys in. We now have the following:

(C) 264, (C#) ?, (D) 297, (D#) ?, (E) 330, (F) 352, (F#) ?, (G) 396, (G#) ?, (A) 440, (A#) ?, (B) 495, (C) 528

So what are the frequencies of the sharps (or flats, if you prefer)? If I look at this as a geometric sequence, I get:

(C) 264, (C#) ?, (D) 297, (D#) ?, (E) 330, (F) 352, (F#) ?, (G) 396, (G#) ?, (A) 440, (A#) ?, (B) 495, (C) 528

t1 **264** t2 \_\_\_ t3 \_\_\_ t4 \_\_\_ t5 \_\_\_ t6 \_\_\_ t7 \_\_\_ t8 \_\_\_ t9 \_\_\_ t10 \_\_\_ t11 \_\_\_ t12 \_\_\_ t13 **528**

So term one ( $t_1$ ) = 264 and the thirteenth term ( $t_{13}$ ) = 528, and the number of terms ( $n$ ) = 13. So putting this into the formula for geometric terms we get:

$$t_n = ar^{n-1}, \text{ so } 528 = 264r^{13-1}$$

$$\frac{528}{264} = r^{12}, \text{ therefore } 2 = r^{12}$$

$$\text{and, } r = \sqrt[12]{2} = 1.059463094$$

So a ratio of 1.059463094 in between each key will allow me to start a triad or a scale on any key, however I do want the tenth term (A) above to equal 440.

So the new problem now looks like this:

t1 \_\_\_ t2 \_\_\_ t3 \_\_\_ t4 \_\_\_ t5 \_\_\_ t6 \_\_\_ t7 \_\_\_ t8 \_\_\_ t9 \_\_\_ t10 **440** t11 \_\_\_ t12 \_\_\_ t13 \_\_\_

So term one ( $t_1$ ) = unknown and the tenth term ( $t_{10}$ ) = 440, and the number of terms ( $n$ ) = 10. However, I now know the ratio ( $r$ ) = 1.059463094 . So putting this into the formula for geometric terms we get:

$$t_n = ar^{n-1}, \text{ so } t_{10} = t_1(\sqrt[12]{2})^{10-1}$$

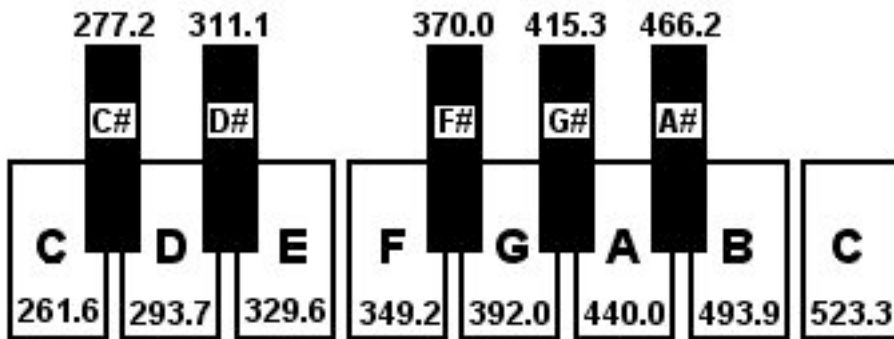
$$440 = t_1(\sqrt[12]{2})^9, \text{ therefore } \frac{440}{(\sqrt[12]{2})^9} = t_1$$

$$\text{and, } t_1 = 261.6255653$$

So, finally, I get the ratios of the modern scale as:

Note	Frequency	Ratio
<b>C</b>	261.6255653	1
<b>C #</b>	277.1826309	1.059463094
<b>D</b>	293.6647677	1.059463094
<b>D #</b>	311.1269834	1.059463094
<b>E</b>	329.6275565	1.059463094
<b>F</b>	349.2282308	1.059463094
<b>F #</b>	369.994422	1.059463094
<b>G</b>	391.9954351	1.059463094
<b>G #</b>	415.3046965	1.059463094
<b>A</b>	440	1.059463094
<b>A #</b>	466.1637599	1.059463094
<b>B</b>	493.8832994	1.059463094
<b>C</b>	523.2511285	1.059463094

And a picture of the modern keyboard, all filled in looks like this:



Doubling and halving these numbers, tripling and dividing by three, and so on, give you the frequencies of all the other octaves.

So starting with the little ratio of **4 : 5 : 6** two days ago, using multiplication and fraction multiplication and our knowledge of geometric sequences (future sessions on this), we have built up the modern musical scale.