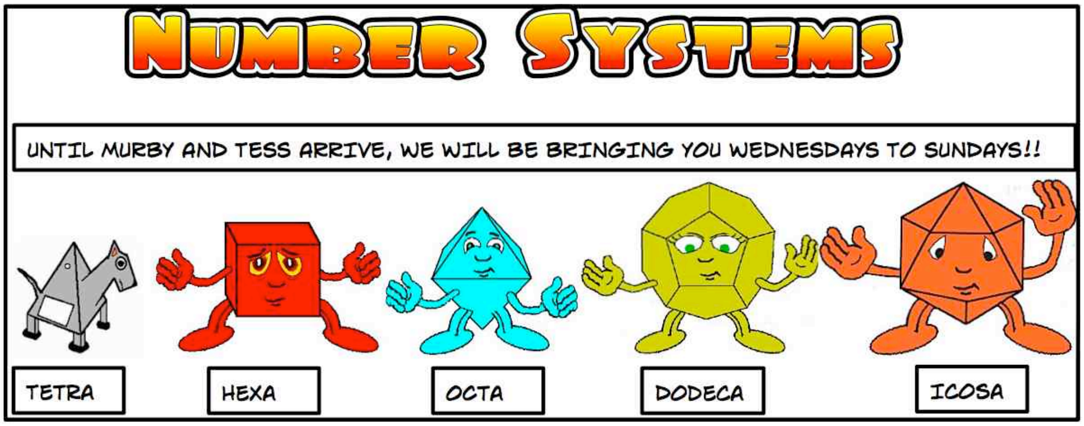
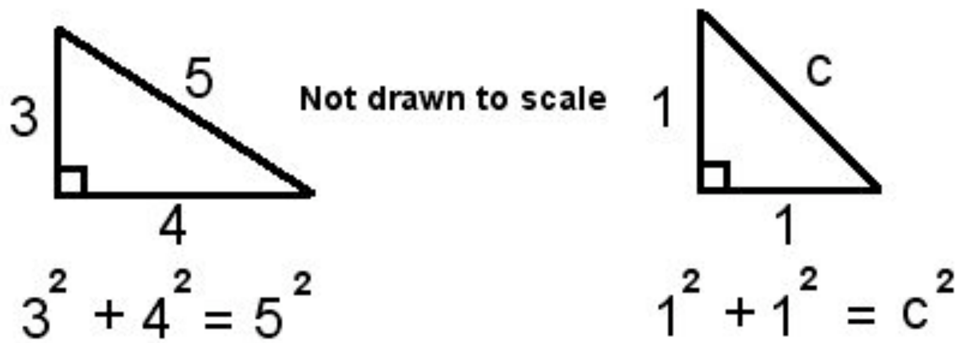


Today's session involves the invention of the last two types of number systems. As we looked at on Monday and Tuesday, these numbers are invented to solve a certain problem. The next problem involved the Pythagoreans (ca. 500 BC) and their wonderful invention of the Pythagorean Theorem: $a^2 + b^2 = c^2$, where "a" and "b" are sides of right triangle having a hypotenuse of "c". The triangle on the left below, illustrates this with the popular 3 – 4 – 5 triangle. However, it was the 1 – 1 – ? triangle on the right that dumbfounded them.



Once they substituted a = 1 and b = 1 into the formula, they got : $1^2 + 1^2 = c^2$, or $1 + 1 = c^2$, or $2 = c^2$. When they went searching for a number, that when multiplied by itself yielded 2, they could not find one. Observing the following: $1.4^2 = 1.96$, too small; $1.5^2 = 2.25$, too big. $1.41^2 = 1.9881$, too small; $1.42^2 = 2.0164$, too big. $1.414^2 = 1.999396$, too small; $1.415^2 = 2.002225$, too big. $1.4142^2 = 1.99996164$, too small; $1.4143^2 = 2.00024449$, too big. And so on, they could not find an answer that was any number that they had invented yet. It was eventually proved by Euclid (ca. 300 BC), that no rational number would ever work (see proof below), and thus the irrational numbers were invented along, eventually, with the new symbol $\sqrt{2}$ that meant "the number that when multiplied by itself is 2". If, a number is not rational, that is, can never be expressed as a fraction in the form of an integer over an integer, then it is irrational. Numbers have to be one or the other, you cannot be both. Other examples of rational numbers that you will bump into while mathhiking are: any non-perfect root $\sqrt{3}, \sqrt[3]{5}, \sqrt[4]{9}$, etc; sin(x), cos(x), tan(x), etc., π (pi), e, the golden ratio, Φ (phi), logarithms and so on. Together the rational numbers and the irrational numbers make up the REAL numbers. Pretty well all the numbers that you will use in high school are Real numbers.

The next invention came about when mathematicians were trying to solve this problem: $x^2 + 1 = 0$. Here is their solution: $x^2 = -1 \Rightarrow x = \sqrt{-1}$ (read the arrow as "therefore"). Now, once again they had a dilemma. No Real number that they had invented could solve this problem. Square root means you must have two EQUAL factors, and 1 x 1 = +1, and -1 x -1 = +1. So there did not exist a number that, when multiplied by itself, equaled

-1. It took Leonhard Euler, pronounced "Oyler" (1707- 1783), to come up with a solution. He invented a new number, (surprise!), the number " i " which he defined as $i^2 = -1$. Thus the imaginary numbers were invented. He also invented the symbol " e " for the base of natural logarithms and " f( ) " for functions. In 1799, Friedrich Gauss (1777 – 1838), proved the fundamental theorem of algebra, that every algebraic equation has a root in the form of a + bi, where a and b are real numbers. When you join a real number with an imaginary number we form the complex number system. If a = 0, we have imaginary numbers, if b = 0 we have the real numbers.

This brings us to the end of our number systems as we know it today. That was a quick hike, starting with natural and whole numbers on Monday, then integers and rational numbers, on Tuesday, and finishing with irrational, real, imaginary and complex numbers on Wednesday. The proof that $\sqrt{2}$ is irrational follows, and soon a poster illustrating all the branches of the number tree will appear in the download section.

Proof that $\sqrt{2}$ is irrational, by Euclid (ca. 300 BC)

Start by assuming that the $\sqrt{2}$ is rational, and can be written as a rational number $\frac{a}{b}$. Since all rational numbers can be reduced, we will assume that $\frac{a}{b}$ is in its lowest form. Thus:

(1) $\sqrt{2} = \frac{a}{b}$, original assumption

(2) $2 = \frac{a^2}{b^2}$, square both sides

(3) $2b^2 = a^2$, multiply both sides by b^2

Therfore a^2 must be even, and "a" must be even

Let a = 2n, for all even numbers are 2 x something

Then (4) $2b^2 = (2n)^2$, substituting a = 2n into (3)

(5) $2b^2 = 4n^2$, squaring out $(2n)^2$

(6) $b^2 = 2n^2$, dividing both sides by 2

Therefore b^2 must be even, and "b" must be even

Therefore $\frac{a}{b}$ is not in its lowest terms,

Since all rational numbers can be put in their lowest terms then our original assumption that $\sqrt{2}$ is rational must be false.

Hence $\sqrt{2}$ is irrational.